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Asymptotic normality of least squares estimators in autoregressive linear regression models

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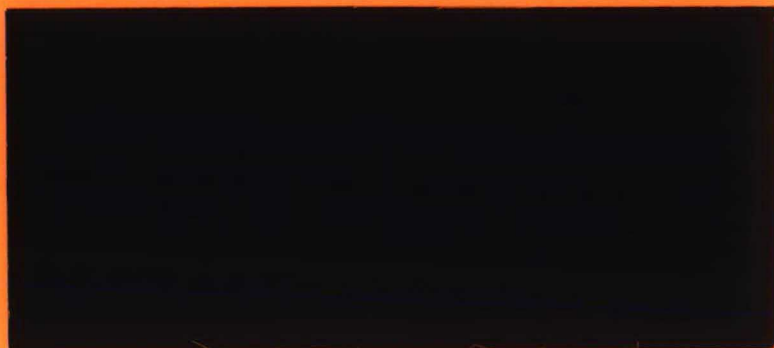
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RESEARCH MEMORANDUM



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ASYMPTOTIC NORMALITY OF LEAST SQUARES
ESTIMATORS IN AUTOREGRESSIVE LINEAR
REGRESSION MODELS

by B.B. van der Genugten

ASYMPTOTIC NORMALITY OF LEAST SQUARES ESTIMATORS IN AUTOREGRESSIVE LINEAR
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For the linear regression model $y_t = \beta' x_t + \varepsilon_t$ ($t = 1, \dots, n$) asymptotic normality of the least squares estimator of β is proved in the case that the ε_t are mutually independent with finite second moments and that ε_t is independent of x_1, \dots, x_t for each t . The results obtained are applied to autoregressive models with nonstochastic, possibly unbounded regressors.

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1. Introduction

In this paper we derive central limit theorems for LS-estimators in linear models with stochastic explanatory variables (regressors). Conditions for these variables bring together the purely deterministic case and the mixed autoregressive case in which some of these variables may be lagged dependent variables.

In the existing literature for the mixed case strong conditions on the behaviour of the regressors are imposed, which are not satisfied for very simple cases. The conditions in this paper are rather weak and cover a wide range of models of practical importance.

The analysis is based on some recently obtained central limit theorems for martingales and some newly developed weak laws of large numbers for linear combinations of random variables. By a special method of normalization the usual but rather unnatural and strong conditions for convergence of the regressors (in some sense) to non-trivial values are circumvented.

We restrict ourselves to univariate models. Generalizations for the multivariate case are easily obtained. So, we shall be concerned with the model

$$(1.1) \quad y_t = \beta' x_t + \varepsilon_t, \quad t \in \mathbb{N}$$

Here, $(x_t)_{t \in \mathbb{N}}$ is a sequence of random $k \times 1$ -vectors of explanatory variables and $(\varepsilon_t)_{t \in \mathbb{N}}$ an error sequence of random variables, both defined on some probability space (Ω, \mathcal{F}, P) . The $k \times 1$ vector β is the nonstochastic vector of regression coefficients.

The assumptions are precisely stated in section 2. We will discuss the main features here.

We assume that the errors are independent and that they are of the same order of magnitude (formally expressed by the Eicker conditions (2.1) below).

The assumptions of independent instead of uncorrelated errors is made to avoid conditions about the existence of moments higher than the second order. Especially for i.i.d. errors assumptions about higher order moments are superfluous and we do not want to lose this special case.

Since the case of lagged dependent variables has to be included we cannot assume that the error process is independent of the regressor process. It is only reasonable to assume that the errors at time t are independent of the regressors before or at time t (conditions (2.2) below). Furthermore we will only consider the (weak) non-collinear case, i.e. the assumptions are always such that they imply $P\{S_n > 0\} \rightarrow 1, n \rightarrow \infty$, where

$$S_n = \sum_{t=1}^n x_t x_t'.$$

A least squares (LS) estimator b_n of β , based on the first n observations, is defined by

$$b_n = S_n^{-1} \sum_{t=1}^n x_t y_t,$$

where S_n^{-1} is some pseudo-inverse of S_n . In the non-collinear case the probability of the set for which S_n^{-1} is not defined tends to zero. So there can be no confusion in writing S_n^{-1} instead of S_n^{-} since we deal with convergence in distribution. Therefore we will do this from now on.

Theorem 2.1 below gives conditions for the x_t and ε_t under which b_n is asymptotically normal, i.e. there exists a sequence C_n of non-stochastic positive definite $k \times k$ matrices, not depending on β , such that

$$\mathcal{L}\{C_n^{-1/2}(b_n - \beta)\} \rightarrow N_k(0, I), \quad n \rightarrow \infty.$$

Here $N_k(0, I)$ denotes the k -dimensional standard normal distribution. For C_n we do not take $\text{Cov}\{S_n^{-1} \sum_{t=1}^n x_t \varepsilon_t\}$ but

$$C_n = \text{Cov}\{\bar{S}_n^{-1} \sum_{t=1}^n x_t \varepsilon_t\},$$

where $\bar{S}_n = E\{S_n\}$. This particular choice admits the use of central limit theorems for martingales. The theorem generalizes the results for non-stochastic x_t of Eicker [4]; theorem 3.1 or [5], section 2, theorem. Furthermore, the conditions are such that they apply to autoregressive models of the form

$$(1.2) \quad \sum_{g=0}^p \alpha_g y_{t-g} = \gamma' w_t + \epsilon_t, \quad t \in \mathbb{N}, \quad (\alpha_0 = 1).$$

Here, $\alpha = (\alpha_1, \dots, \alpha_p)'$ and γ are nonstochastic $p \times 1$ and $q \times 1$ vectors, respectively, of regression coefficients, w_t is a nonstochastic $q \times 1$ vector, and the initial values y_{1-p}, \dots, y_0 are random variables. By taking $\beta' = (\gamma', -\alpha')$ and $x_t' = (w_t', y_{t-1}, \dots, y_{t-p})$ we see that (1.2) is a special case of (1.1).

Theorem 2.2 below states that for this autoregressive model the conditions for the x_t in theorem 2.1 are implied by certain conditions for the w_t and the α, γ . The proof of this result forms the hard and technical part of the paper. The result is attractive because in these conditions the α, γ appear in a simple way. In particular there is no need to solve the difference equation (1.2) explicitly.

Finally, theorem 2.3 below is added to simplify the verification of the conditions of theorem 2.2. The result in the first part of example 2.2 given after this theorem is simpler but can be compared with results in the literature up till now such as Anderson [1], theorem 5.5.14 or Schönfeld [10]. The example 2.3 there after shows clearly the advantages of the methods used in this paper. Simple practical problems with unbounded (w_t) can be solved too. So we can say that an old problem which goe back to Mann and Wald [8] has found a satisfactory solution now in the theorems 2.1 and 2.2.

From a statistical point of view it is an interesting question if C_n can be estimated while preserving asymptotic normality of b_n . This can be done along the lines of Eicker [4]. Here σ_t^2 is replaced by the square of the least-squares-residual at t . We will not go into details.

2. Statement of the results

In the following we suppose that the ε_t are mutually independent with $E\{\varepsilon_t\} = 0$, $\sigma_t^2 = E\{\varepsilon_t^2\} < \infty$, $t \in \mathbb{N}$. Furthermore, we assume that they satisfy the Eicker conditions (see e.g. Eicker [4], theorem 3.1):

$$(2.1) \quad 1 := \inf_{(t \in \mathbb{N})} \sigma_t^2 > 0, \quad \sup_{(t \in \mathbb{N})} E\{\varepsilon_t^2 \mid |\varepsilon_t| \geq \delta\} \rightarrow 0, \quad \delta \rightarrow \infty.$$

In the following conditions it is assumed that $\bar{S}_n > 0$ for some (and therefore all) n sufficiently large whenever \bar{S}_n^{-1} appears.

Theorem 2.1 (general model (1.1)). If

$$(2.2) \quad \varepsilon_t \text{ is independent of } x_1, \dots, x_t \text{ and } E|x_t|^2 < \infty \text{ for each } t \in \mathbb{N},$$

$$(2.3) \quad \bar{S}_n^{-\frac{1}{2}} \left(\sum_{t=1}^n \lambda_t x_t x_t' - E \left\{ \sum_{t=1}^n \lambda_t x_t x_t' \right\} \right) \bar{S}_n^{-\frac{1}{2}} \xrightarrow{P} 0$$

$$\text{for } \lambda_t = 1 \text{ and } \lambda_t = \sigma_t^2,$$

$$(2.4) \quad \max_{(1 \leq t \leq n)} x_t' \bar{S}_n^{-1} x_t \xrightarrow{P} 0,$$

then

$$\mathcal{L}\{C_n^{-\frac{1}{2}}(b_n - \beta)\} \rightarrow N_k(0, I).$$

Remark. The condition (2.3) for $\lambda_t = 1$ gives $\bar{S}_n^{-\frac{1}{2}} S_n \bar{S}_n^{-\frac{1}{2}} \xrightarrow{P} I$, implying weak noncollinearity.

Example 2.1 (nonstochastic x_t). For such x_t the conditions (2.2), (2.3) are fulfilled in a trivial way and (2.4) reduces to

$$\max_{(1 \leq t \leq n)} x_t' S_n^{-1} x_t \rightarrow 0$$

or, equivalently (lemma 3.5 below),

$$S_n^{-1} \rightarrow 0, \quad x_n' S_n^{-1} x_n \rightarrow 0.$$

This implies that x_n is non-exponentially increasing (i.e. $\rho^n x_n \rightarrow 0$ for any ρ with $|\rho| < 1$). However, polynomial trends are included. For a discussion of the necessity of these conditions and for some special cases see Eicker [5].

For the general model (1.1) the appearance of (2.4) in theorem 2.1 is motivated by example 2.1. The condition (2.3) for $\lambda_t = \sigma_t^2$ admits the use of a central limit theorem for martingales.

For the model (1.2) we assume that the starting values y_{1-p}, \dots, y_0 are independent of $(\varepsilon_t)_{t \in \mathbb{N}}$ and have finite second moments. Let the $(p+q) \times 1$ vector v_t be defined by

$$v_t' = \left(\sum_{g=0}^p \alpha_g w_{t-g}', \gamma' w_{t-1}', \dots, \gamma' w_{t-p}' \right), \quad t = p+1, p+2, \dots$$

and the $(p+q) \times (p+q)$ matrix Z_n by

$$Z_n = \sum_{t=p+1}^n v_t v_t' + n I_0.$$

Here I_0 is a $(p+q) \times (p+q)$ matrix with all elements equal to zero except the last p elements of its diagonal which are equal to one. Finally, let

$$A(z) = \sum_{g=0}^p \alpha_g z^g, \quad z \in \mathbb{C}.$$

We have:

Theorem 2.2 (autoregressive model (1.2)). If

$$(2.5) \quad A(z) \neq 0, \quad |z| \leq 1,$$

$$(2.6) \quad \|Z_n^{-1}\| = O(1/n),$$

$$(2.7) \quad v_n' Z_n^{-1} v_n \rightarrow 0,$$

then the conditions (2.2) - (2.4) of theorem 2.1 are satisfied.

For the model (1.2) the condition (2.5) is the usual stability condition. The conditions (2.6) and (2.7) are more difficult to explain. Roughly spoken, in some sense, v_t can serve as an approximation for $\mu_t = E\{x_t\}$. Then $\Sigma v_t v_t'$ can be considered to be an approximation of the part of \bar{S}_n coming from the w_t . The other part of \bar{S}_n coming from products of errors is approximated by nI_0 . So Z_n is an approximation of \bar{S}_n . In fact we have that $cZ_n \leq \bar{S}_n \leq c'Z_n$ for some $c, c' > 0$ and n sufficiently large. So we may hope that (2.4) can be replaced by the condition $\max v_t' Z_n^{-1} v_t \rightarrow 0$ or equivalently, using lemma 3.5, b), $Z_n^{-1} \rightarrow 0$ and (2.7). The condition (2.6) is slightly stronger, and is needed to justify the approximations as well as to verify the weak law of large numbers as stated in (2.3).

For many interesting sequences (w_t) the verification of the conditions (2.6), (2.7) of theorem 2.2 is not easy. Therefore we add theorem 2.3 below to simplify this verification.

Suppose for some integer $s \geq q$ there exists a sequence $(q_t)_{t \geq p+1}$ of nonstochastic $s \times 1$ vectors such that

$$v_t = \Phi q_t, \quad t = p + 1, p + 2, \dots$$

for some nonstochastic $(p+q) \times s$ matrix Φ . Let $(D_n)_{n \geq p+1}$ be a sequence of $s \times s$ nonstochastic positive definite diagonal matrices. Consider the normed $s \times 1$ vectors

$$\tilde{q}_t(n) = D_n^{-1/2} q_t, \quad t = p + 1, \dots, n.$$

For some integer r such that $q \leq r \leq \min(s, p + q)$ let ϕ_{00} be the $r \times r$ submatrix of Φ formed by the first r rows and r columns of Φ , let D_{n0} be a similar $r \times r$ submatrix of D_n , and for the case that $r < s$ let D_{n1} be the $(s-r) \times (s-r)$ submatrix of D_n formed by the last $s - r$ rows and $s - r$ columns of D_n . We have:

Theorem 2.3. Suppose

$$(2.8) \quad \det(\phi_{00}) \neq 0,$$

$$(2.9) \quad \|D_{n0}^{-1}\| = o(1/n),$$

$$(2.10) \quad \text{if } r < s \text{ then } \|D_{n1}\| = o(1/\|D_{n0}^{-1}\|),$$

$$(2.11) \quad \left\| \left(\sum_{p+1}^n \tilde{q}_t(n) \tilde{q}_t'(n) \right)^{-1} \right\| = o(1),$$

$$(2.12) \quad \tilde{q}_n(n) \rightarrow 0,$$

then the conditions (2.6), (2.7) of theorem 2.2 are satisfied.

Example 2.2. We apply theorem 2.3 for the simple case $s = (p+1)q$ and $q_t' = (w_t', \dots, w_{t-p}')$. Then $v_t = \phi q_t$ with

$$\phi = \begin{bmatrix} \alpha_0 I & \alpha_1 I & \dots & \alpha_p I \\ 0 & \gamma' & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \gamma' \end{bmatrix}$$

Take $D_n = nI$. Then the conditions (2.11), (2.12) are satisfied if

$$(2.13) \quad \left\| \left(\sum_{p+1}^n q_t q_t' \right)^{-1} \right\| = o(1/n), \quad q_n = o(\sqrt{n})$$

Take $r = q$. Then (2.9), (2.10) are fulfilled. Since $\phi_{00} = \alpha_0 I$ the condition (2.8) holds as well. Therefore, with theorem 2.3 we see that (2.13) leads to the asymptotic normality of the LS-estimator in the autoregressive model (1.2) for all γ and all α satisfying (2.5). The condition (2.13) is implied by

$$\frac{1}{n} \sum_{p+1}^n q_t q_t' \rightarrow Q$$

for some $Q > 0$.

Note that (2.13) will not be satisfied if the model contains a constant term. However, a slight modification of the derivation above leads again to a condition of the type (2.13). We write $w'_t = (1, \tilde{w}'_t)$, $\gamma' = (\gamma_0, \tilde{\gamma}')$. Choose $s = pq + q - p$ and $q'_t = (1, \tilde{w}'_t, \dots, \tilde{w}'_{t-p})$. Then $v_t = \phi q_t$ with

$$\phi = \begin{bmatrix} p & & & & \\ \sum \alpha_g & 0 & 0 & \dots & 0 \\ 0 & \alpha_g & & & \\ 0 & \alpha_0 I & \alpha_1 I & \dots & \alpha_p I \\ \gamma_0 & 0 & \tilde{\gamma}' & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_0 & 0 & 0 & \dots & \tilde{\gamma}' \end{bmatrix}$$

Take $r = q$ then again $\det(\phi_{00}) \neq 0$ since $\sum_0^p \alpha_g \neq 0$ because of (2.5).

Example 2.3. Let $w'_t = (t^{a_1}, \dots, t^{a_q})$ for integers $a_1 > a_2 > \dots > a_q \geq 0$. If $a_1 > 0$ then (2.13) of example 2.2 cannot be applied. For $q \leq r \leq \min(a_1 + 1, p + q)$ we can write

$$\begin{pmatrix} w_t \\ w_{t-1} \\ \vdots \\ w_{t-p} \end{pmatrix} = \begin{bmatrix} \psi_0 & * \\ \psi_1 & * \\ \vdots & \\ \psi_p & * \end{bmatrix} \begin{pmatrix} t^{a_1} \\ t^{a_1-1} \\ \vdots \\ 1 \end{pmatrix}$$

Here ψ_0, \dots, ψ_p are $q \times r$ upper triangular matrices. Not interesting matrices are denoted by stars. Choose $s = a_1 + 1$ and $q'_t = (t^{a_1}, t^{a_1-1}, \dots, 1)$. Then $v_t = \phi q_t$ with

$$\phi = \begin{bmatrix} \alpha_0 I & \alpha_1 I & \dots & \alpha_p I \\ 0 & \gamma' & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & \gamma' \end{bmatrix} \begin{bmatrix} \psi_0 & * \\ \psi_1 & * \\ \vdots & \vdots \\ \psi_p & * \end{bmatrix} \Rightarrow \phi_{00} = \begin{bmatrix} p \\ \sum \alpha_j \psi_j \\ 0 \\ \gamma' \psi_1 \\ \vdots \\ \gamma' \psi_{r-q} \end{bmatrix}$$

Take $D_n = \text{diag}(n^{2a_1+1}, n^{2a_1-1}, \dots, n)$. Then

$$\sum_{t=1}^n \tilde{q}_t(n) \tilde{q}_t'(n) \rightarrow H$$

where $H = \{(2a_1+3-i-j)^{-1}; i, j = 1, \dots, a_1+1\}$ is the so-called Hilbert-matrix. Note that $H > 0$ and $\tilde{q}_t(n) \rightarrow 0$ for fixed t . Therefore the condition (2.11) is satisfied. Since

$$\tilde{q}_n(n) = D_n^{-1/2} q_n = n^{-1/2} (1, 1, \dots, 1)'$$

we see that (2.12) holds. Furthermore,

$$D_{n0} = \text{diag}(n^{2a_1+1}, \dots, n^{2a_1+3-2r})$$

$$D_{n1} = \text{diag}(n^{2a_1+1-2r}, \dots, n) \text{ if } r < a_1 + 1,$$

and so the conditions (2.9), (2.10) are fulfilled also. Therefore it remains to verify the condition (2.8). In general this cannot be done for all γ and all α satisfying (2.5):

1^o) Consider the case of successive powers

$$a_q = a_1 - q + 1$$

Then all ψ_j have diagonal elements 1. From (2.5) it follows that $\sum_{j=0}^p \alpha_j \neq 0$. Hence, if we take $r=q$ then $\phi_{00} = \sum_{j=0}^p \alpha_j \psi_j$ is nonsingular, and consequently the condition (2.8) is fulfilled for all γ and all α satisfying (2.5).

2^o) Consider the particular case of non-successive powers $p = 1, q = 2, a_1 = 2, a_2 = 0$. Set $\gamma' = (\gamma_1, \gamma_2)$. The choice $r = 3$ leads to

$$\phi_{00} = \begin{bmatrix} \alpha_0 + \alpha_1 & -2\alpha_1 & \alpha_1 \\ 0 & 0 & \alpha_0 + \alpha_1 \\ \gamma_1 & -2\gamma_1 & \gamma_1 + \gamma_2 \end{bmatrix} \Rightarrow \det(\phi_{00}) = 2\alpha_0\gamma_1(\alpha_0 + \alpha_1).$$

So the condition (2.8) is fulfilled for all γ with $\gamma_1 \neq 0$ and for all α satisfying (2.5). This makes clear that in this example it is difficult to prove the asymptotic normality for all γ and all α satisfying (2.5).

A special case of example 2.3, 1^0) is the model

$$y_t + \alpha_1 y_{t-1} = \gamma_1 + \gamma_2 t + \varepsilon_t$$

which can serve as an empirical description of the log of the Dutch national income during the years 1949-1968. As far as I know the available asymptotic results on literature to justify the usual estimation and testing procedures are not applicable to this simple model. This was a part of the motivation to write this paper.

3. Proofs of the theorems

We mention the following notations and conventions. For the norm $\|A\|$ of an arbitrary matrix A we take $\lambda_{\max}^{1/2}(A'A)$. We write $A \geq 0$ if A is positive semi-definite and $A > 0$ if A is positive definite. We write $A \geq B$ if $A - B \geq 0$ and $A - B > 0$. Note that $A \geq 0$ implies $\|A\| = \lambda_{\max}(A)$ and $A > 0$ that $\|A^{-1}\| = \lambda_{\min}^{-1}(A)$.

The random sequence $(x_n)_{n \in \mathbb{N}}$ is called P -bounded if for every $\varepsilon > 0$ there exists a number M such that $P\{|x_n| \geq M\} < \varepsilon$ for all n . If $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are sequences of random variables such that (x_n) is P -bounded and $y_n \xrightarrow{P} 0$ then $x_n y_n \xrightarrow{P} 0$. If $E|x_n|$ is bounded then (x_n) is P -bounded.

The proof of theorem 2.1 is based on a central limit theorem for martingale triangular arrays. Let

$$1 = \inf_{(t \in \mathbb{N})} \sigma_t^2, \quad m = \sup_{(t \in \mathbb{N})} \sigma_t^2,$$

then (2.1) implies

$$(3.1) \quad 0 < 1 \leq m < \infty$$

Proof of theorem 2.1. We take

$$C_n = \text{Cov}\{\bar{S}_n^{-1} \sum_{t=1}^n x_t \varepsilon_t\}.$$

Then C_n does not depend on β . We have

$$\begin{aligned} C_n^{-1/2} (b_n - \beta) &= C_n^{-1/2} \bar{S}_n^{-1} \sum_{t=1}^n x_t \varepsilon_t = \\ &= [I + C_n^{-1/2} \bar{S}_n^{-1/2} (\bar{S}_n^{-1/2} \bar{S}_n^{-1} \bar{S}_n^{-1/2} - I) \bar{S}_n^{-1/2} C_n^{1/2}] \cdot C_n^{-1/2} \bar{S}_n^{-1} \sum_{t=1}^n x_t \varepsilon_t. \end{aligned}$$

From (2.2) and (3.1) we get

$$(3.2) \quad 1 \bar{S}_n^{-1} \leq C_n = \bar{S}_n^{-1} E\{\sum_{t=1}^n \sigma_t^2 x_t x_t'\} \bar{S}_n^{-1} \leq m \bar{S}_n^{-1}$$

This implies that $\|\bar{S}_n^{-1/2} C_n^{1/2}\|$ and $\|C_n^{-1/2} \bar{S}_n^{-1/2}\|$ are bounded. From (2.3) for $\lambda_t = 1$ it follows that $\bar{S}_n^{-1/2} S_n \bar{S}_n^{-1/2} \xrightarrow{P} I$. Therefore in the relation above (3.2) the expression between square brackets tends to I in probability. Thus it remains to prove that

$$\mathcal{L}\left\{C_n^{-1/2} \bar{S}_n^{-1} \sum_{t=1}^n x_t \varepsilon_t\right\} \rightarrow N_K(0, I).$$

Let $a \in \mathbb{R}_K$ with $|a| = 1$ and set $x_{nt} = a' C_n^{-1/2} \bar{S}_n^{-1} x_t \varepsilon_t$. Then it suffices to prove

$$(3.3) \quad \mathcal{L}\left\{\sum_{t=1}^n x_{nt}\right\} \rightarrow N(0, 1).$$

Let $\mathcal{F}_{n0} = \mathcal{F}(x_1)$, $\mathcal{F}_{nt} = \mathcal{F}(x_1, \dots, x_{t+1}, \varepsilon_1, \dots, \varepsilon_t)$ for $1 \leq t \leq n$. Then the \mathcal{F}_{nt} are sub- σ -fields of \mathcal{F} with $\mathcal{F}_{n,t-1} \subset \mathcal{F}_{nt}$. Furthermore, x_{nt} is \mathcal{F}_{nt} -measurable and ε_t is independent of $\mathcal{F}_{n,t-1}$, as follows from (2.2). This gives $E\{x_{nt} | \mathcal{F}_{n,t-1}\} = 0$ a.s., and so $(x_{nt})_{1 \leq t \leq n, n \in \mathbb{N}}$ is a martingale triangular array (MTA). Furthermore,

$$\sum_{t=1}^n E\{x_{nt}^2 | \mathcal{F}_{n,t-1}\} - 1 = a' C_n^{-1/2} \bar{S}_n^{-1} \left(\sum_{t=1}^n \sigma_t^2 x_t x_t' - E\left\{\sum_{t=1}^n \sigma_t^2 x_t x_t'\right\} \right) \bar{S}_n^{-1} C_n^{-1/2} a \xrightarrow{P} 0.$$

using (2.3) for $\lambda_t = \sigma_t^2$ and the boundedness of $\|C_n^{-1/2} \bar{S}_n^{-1/2}\|$.

Then (3.3) follows from a central limit theorem for MTA's (see e.g. Mcleish [9], Corollary (3.8) or Gaenssler [6], theorem 2), provided that we can verify the conditional Lindeberg-Feller condition.

$$(3.4) \quad \sum_{t=1}^n E\{x_{nt}^2 \mathbf{I}(|x_{nt}| \geq \varepsilon) | \mathcal{F}_{n,t-1}\} \xrightarrow{P} 0, \text{ for every } \varepsilon > 0.$$

Set

$$r_{nt} = x_t' \bar{S}_n^{-1} x_t, \quad r_n = \max_{(1 \leq t \leq n)} r_{nt}$$

$$Q_t(\delta) = E\{\varepsilon_t^2 \mathbf{I}(|\varepsilon_t| \geq \delta)\}, \quad R_n(\delta) = \max_{(1 \leq t \leq n)} Q_t(\delta)$$

Then (2.3) gives

$$x_{nt}^2 \leq x_t' \bar{S}_n^{-1} C_n^{-1} \bar{S}_n^{-1} x_t \cdot \varepsilon_t^2 \leq (r_{nt}/m) \cdot \varepsilon_t^2.$$

Since r_{nt} is $\mathcal{F}_{n,t-1}$ -measurable and ε_t is independent of $\mathcal{F}_{n,t-1}$ it follows that

$$\begin{aligned} & \sum_{n=1}^{\infty} E\{x_{nt}^2 \mathbf{I}(|x_{nt}| \geq \varepsilon) \mid \mathcal{F}_{n,t-1}\} \leq \\ & \leq \sum_{n=1}^{\infty} r_{nt} E\{\varepsilon_t^2 \mathbf{I}(|\varepsilon_t| \geq \varepsilon \sqrt{m/r_{nt}}) \mid \mathcal{F}_{n,t-1}\} = \\ & = \sum_{n=1}^{\infty} r_{nt} Q_t(\varepsilon \sqrt{m/r_{nt}}) \leq \sum_{n=1}^{\infty} r_{nt} Q_t(\varepsilon \sqrt{m/r_n}) \leq \\ & \leq R_n(\varepsilon \sqrt{m/r_n}) \cdot \sum_{n=1}^{\infty} r_{nt}. \end{aligned}$$

From (2.1) we get that $R_n(\delta_n) \rightarrow 0$ for any nonstochastic sequence (δ_n) with $\delta_n \rightarrow \infty$, and from (2.4) that $r_n \xrightarrow{P} 0$. By considering the a.s. convergence of subsequences it follows that $R_n(\varepsilon \sqrt{m/r_n}) \xrightarrow{P} 0$. Furthermore, $E\{\sum_{n=1}^{\infty} r_{nt}\} = k$ and $r_{nt} \geq 0$ imply that $\sum_{n=1}^{\infty} r_{nt}$ is P -bounded. Together this shows that the right-hand side of (3.5) tends to 0 in probability, completing the proof.

The proof of theorem 2.2 is rather tedious. We make some preliminary remarks and formulate some intermediate results as lemma's. Let

$$m_n = \max_{(1 \leq t \leq n)} \sigma_t^2, \quad s_n = \sum_{t=1}^n \sigma_t^2$$

From (2.1) it follows that $s_n \geq n1$ and that for every $\varepsilon > 0$

$$(3.6) \quad s_n^{-1} \sum_{t=1}^n E\{\varepsilon_t^2 \mathbf{I}(|\varepsilon_t| \geq \varepsilon \sqrt{s_n})\} \leq 1^{-1} \sup_{t \in \mathbb{N}} E\{\varepsilon_t^2 \mathbf{I}(|\varepsilon_t| \geq \varepsilon \sqrt{nl})\} \rightarrow 0.$$

So the Eicker conditions (2.1) imply the well-known Lindeberg-Feller (LF) condition for the sequence $(\varepsilon_t)_{t \in \mathbb{N}}$. This leads to the corollary of lemma 3.1 below. The lemma itself is a slight generalization of a theorem of Raikov (see Gnedenko [7], § 28, theorem 4). Its proof is kept short and added for the sake of completeness.

Lemma 3.1. Let $(x_{nj})_{1 \leq j \leq k_n, n \in \mathbb{N}}$ be a triangular array of random variables such that x_{n1}, \dots, x_{nk_n} are mutually independent for each fixed n .

Suppose $E\{x_{nj}\} = 0$, $\sigma_{nj}^2 = E\{x_{nj}^2\} < \infty$ for all n, j .

If the LF condition

$$\sum_{j=1}^{k_n} E\{x_{nj}^2 I(|x_{nj}| \geq \varepsilon)\} \rightarrow 0, \text{ for every } \varepsilon > 0,$$

holds, and if $\sum_{j=1}^{k_n} \sigma_{nj}^2$ is bounded in n , then

$$\max_{(1 \leq j \leq k_n)} |x_{nj}| \xrightarrow{P} 0$$

$$\max_{(1 \leq j \leq k_n)} \sigma_{nj}^2 \rightarrow 0,$$

and for any bounded nonstochastic array (λ_{nj}) we have

$$\sum_{j=1}^{k_n} \lambda_{nj} (x_{nj}^2 - \sigma_{nj}^2) \xrightarrow{P} 0.$$

Proof. The first relation follows from

$$P\{\max_{1 \leq j \leq k_n} |x_{nj}| \geq \varepsilon\} \leq \sum_{j=1}^{k_n} P\{|x_{nj}| \geq \varepsilon\} \leq \varepsilon^{-2} \sum_{j=1}^{k_n} E\{x_{nj}^2 I(|x_{nj}| \geq \varepsilon)\}$$

and the second one from

$$\sigma_{nj}^2 \leq \varepsilon^2 + \max_{1 \leq j \leq k_n} E\{x_{nj}^2 I(|x_{nj}| \geq \varepsilon)\} \leq \varepsilon^2 + \sum_{j=1}^{k_n} E\{x_{nj}^2 I(|x_{nj}| \geq \varepsilon)\}.$$

The last relation will be proved first for $\lambda_{nj} = 1$.

Let $\varphi_{nj}(u) = E\{\exp(iu x_{nj})\}$ then it suffices to prove that

$$\sum_{j=1}^{k_n} (\log \varphi_{nj}(u) - iu \sigma_{nj}^2) \rightarrow 0. \text{ Since } |\varphi_{nj}(u) - 1| \leq |u| \sigma_{nj}^2 \text{ we have}$$

$$\max_{(1 \leq j \leq k_n)} |\varphi_{nj}(u) - 1| \rightarrow 0, \quad \sum_{j=1}^{k_n} |\varphi_{nj}(u) - 1|^2 \rightarrow 0$$

Since $|\log(1+z) - z| \leq |z|^2$, $|z| \leq 1/2$, this gives

$$\sum_{j=1}^{k_n} |\log \varphi_{nj}(u) + 1 - \varphi_{nj}(u)| \rightarrow 0$$

From

$$|\varphi_{nj}(u) - 1 - iu\sigma_{nj}^2| \leq \frac{1}{2} |u|^2 \varepsilon \sigma_{nj}^2 + 2|u| E\{x_{nj}^2 \mathbf{I}(x_{nj}^2 \geq \varepsilon)\}$$

we obtain

$$\sum_1^{k_n} |\varphi_{nj}(u) - 1 - iu\sigma_{nj}^2| \rightarrow 0$$

Together this gives $\sum_1^{k_n} (\log \varphi_{nj}(u) - iu\sigma_{nj}^2) \rightarrow 0$, proving the result for $\lambda_{nj} = 1$.

For non-negative λ_{nj} the result follows from this. Replace only x_{nj} by $\lambda_{nj}^{1/2} x_{nj}$.

For arbitrary λ_{nj} the result follows by splitting up the sum for positive and negative λ_{nj} .

Corollary. By taking $x_{nj} = \varepsilon_j / \sqrt{s_n}$ we see that the LF-condition for (x_{nj}) follows from (3.6). Hence,

$$(3.7) \quad s_n^{-1/2} \max_{(1 \leq t \leq n)} |\varepsilon_t| \xrightarrow{P} 0$$

$$(3.8) \quad m_n / s_n \rightarrow 0$$

and for bounded nonstochastic $(\lambda_{nt})_{1 \leq t \leq n, n \in \mathbb{N}}$:

$$(3.9) \quad s_n^{-1} \sum_1^n \lambda_{nt} (\varepsilon_t^2 - \sigma_t^2) \xrightarrow{P} 0$$

Lemma 3.2. For nonstochastic φ_h and ψ_h ($h = 0, 1, \dots$) let

$$y_t = \sum_{h=0}^{\infty} \varphi_h \varepsilon_{t-h}, \quad z_t = \sum_{h=0}^{\infty} \psi_h \varepsilon_{t-h}, \quad t \in \mathbb{Z},$$

where $\varepsilon_t = 0$ if $t \leq -p$, $\sigma_t^2 = E\{\varepsilon_t^2\} < \infty$ if $1-p \leq t \leq 0$.

If $\sum_{h=0}^{\infty} \varphi_h z^h$, $\sum_{h=0}^{\infty} \psi_h z^h$, have convergence radii larger than 1, then for any bounded nonstochastic array $(\lambda_{nt})_{1 \leq t \leq n, n \in \mathbb{N}}$ we have

$$(3.10) \quad s_n^{-1} \left(\sum_{t=1}^n \lambda_{nt} y_{t-i} z_{t-j} - E \left\{ \sum_{t=1}^n \lambda_{nt} y_{t-1} z_{t-j} \right\} \right) \xrightarrow{P} 0, \quad i, j \in \mathbb{Z}$$

Proof. Note that $\varphi_h = o(\rho^h)$, $\psi_h = o(\rho^h)$ for some ρ with $0 < \rho < 1$. Set $\varphi_r = \psi_r = 0$ if $r < 0$. Then

$$(3.11) \quad \sum_{t=1}^n \lambda_{nt} y_{t-i} z_{t-j} = \sum_t \lambda_{nt} \left(\sum_r \varphi_r \varepsilon_{t-r-i} \sum_s \psi_s \varepsilon_{t-j-s} \right) = \sum_{r,s} \sum_{nrs} a_{nrs} \varepsilon_r \varepsilon_s$$

where

$$a_{nrs} = \sum_t \lambda_{nt} \varphi_{t-i-r} \psi_{t-j-s}$$

Note that $a_{nrs} = 0$ if $r > n - i$ or $s > n - j$. For any $a \geq b$ we have

$$\left| \sum_{t=1}^n \lambda_{nt} \varphi_{t-a} \psi_{t-b} \right| \leq c' \cdot \sum_{t=a}^n |\varphi_{t-a} \psi_{t-b}| \leq c'' \cdot \sum_{t=a}^n \rho^{t-a} \rho^{t-b} \leq c''' \cdot \rho^{a-b}$$

for some constants c' , c'' , c''' . Therefore there exists a constant c not depending on n, r, s such that

$$|a_{nrs}| \leq c \rho^{|r-s|}.$$

In particular, a_{nrs} is bounded in n, r, s . We split up

$$\sum_{r,s} a_{nrs} \varepsilon_r \varepsilon_s = \sum_{k=1}^5 A_k(n)$$

according to the ranges $(r, s \leq 0)$, $(r \geq 1, s \leq 0)$, $(r \leq 0, s \geq 1)$, $(r, s \geq 1 \text{ and } r \neq s)$, $(r, s \geq 1 \text{ and } r = s)$, respectively.

The number of terms in $A_1(n)$ is finite and the terms themselves are bounded in n . Furthermore,

$$E|A_2(n)| < \sum_{s \leq 0} \sum_{r=1}^{n-1} |a_{nrs}| E|\varepsilon_s \varepsilon_r| \leq c' \cdot \sum_{s \leq 0} \sum_{r=1}^{n-1} \rho^{r-s} \sigma_s \sigma_r \leq c'' \sqrt{m_{n-1}},$$

for some constants c' , c'' . A similar relation holds for $A_3(n)$. Hence, with (3.8) we get

$$(3.12) \quad s_n^{-1} E|A_k(n)| \rightarrow 0, \quad k = 1, 2, 3.$$

Since $E\{A_4(n)\} = 0$ and

$$\begin{aligned} V\{A_4(n)\} &= \sum_{r \neq s} \sum a_{nrs} (a_{nrs} + a_{nsr}) \sigma_r^2 \sigma_s^2 \leq 2c^2 \sum_{r \neq s} \rho^2 |r-s| \sigma_r^2 \sigma_s^2 = \\ &= 2c^2 \left(\sum_{r > s} \sum \rho^{2(s-r)} + \sum_{s > r} \sum \rho^{2(r-s)} \right) \sigma_r^2 \sigma_s^2 \leq \\ &\leq c' (m_{n-j} s_{n-i} + m_{n-i} s_{n-j}) \end{aligned}$$

for some c' , we get with (3.8) that

$$(3.13) \quad s_n^{-1} (A_4(n) - E\{A_4(n)\}) \xrightarrow{P} 0.$$

Finally, with (3.9) we get

$$(3.14) \quad s_n^{-1} (A_5(n) - E\{A_5(n)\}) = s_n^{-1} \sum_{t=1}^{n-\max(i,j)} a_{ntt} (\epsilon_t^2 - \sigma_t^2) \xrightarrow{P} 0.$$

Combining (3.11) - (3.14) gives (3.10).

Lemma 3.3. Let $(x_t)_{t \in \mathbb{N}}$ be a sequence of $k \times 1$ vectors and $(\varphi_h)_{h \in \mathbb{Z}}$ a sequence of scalars. Let

$$y_{nj} = \sum_{t=1}^n x_t \varphi_{t-j}, \quad n \in \mathbb{N}, \quad j \in \mathbb{Z}$$

If $\sum |\varphi_h| < \infty$ then

$$\sum_j y_{nj} y'_{nj} \leq c \cdot \sum_{t=1}^n x_t x'_t, \quad n \in \mathbb{N}$$

for some constant c not depending on n .

Proof. Set

$$\varphi(\lambda) = \sum_j \varphi_j e^{i\lambda j}, \quad x_n(\lambda) = \sum_{t=1}^n x_t e^{it\lambda}, \quad y_n(\lambda) = \sum_j y_{nj} e^{ij\lambda}$$

then $Y_n(\lambda) = X_n(\lambda)\varphi(-\lambda)$. With

$$c = \sup_{(|\lambda| < \pi)} |\varphi(1-\lambda)|^2$$

this gives

$$\begin{aligned} \sum_j y_{nj} y'_{nj} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} Y_n(\lambda) Y_n^*(\lambda) d\lambda = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi(-\lambda)|^2 X_n(\lambda) X_n^*(\lambda) d\lambda \leq \\ &\leq c \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} X_n(\lambda) X_n^*(\lambda) d\lambda = c \cdot \sum_1^n x_t x'_t \end{aligned}$$

Lemma 3.4. Let $(x_t)_{t \in \mathbb{Z}}$ be a sequence of $k \times 1$ vectors with $x_t = 0$ if $t \leq -p$ for some $p \geq 0$, and let $(\varphi_h)_{h \geq 0}$ be a sequence of scalars. Let

$$y_t = \sum_{h=0}^{\infty} \varphi_h x_{t-h}, \quad t \in \mathbb{Z}.$$

If $\varphi(z) = \sum_0^{\infty} \varphi_h z^h$ has convergence radius larger than 1 and $\varphi(z) \neq 0$ if $|z| \leq 1$, then there exist constants $c_1, c_2 > 0$, only depending on (φ_h) , such that

$$c_1 \sum_{t=1-p}^n x_t x'_t \leq \sum_{t=1-p}^n y_t y'_t \leq c_2 \sum_{t=1-p}^n x_t x'_t, \quad n \in \mathbb{N}.$$

Proof. Let $A(z) = 1/\varphi(z) = \sum_0^{\infty} \alpha_h z^h$, then $\sum_0^h \alpha_m \varphi_{h-m} = \delta_{0h}$ for $h = 0, 1, \dots$

Note that

$$\sum_0^{\infty} |\varphi_h| < \infty \quad \sum_0^{\infty} |\alpha_h| < \infty.$$

Let L_n be the $(n+p) \times (n+p)$ lag matrix with unit elements just below the diagonal and zero elements elsewhere. Then $L_n^m \cdot [x_{1-p}, \dots, x_n]' =$

$[x_{1-p-m}, \dots, x_{n-m}]'$ with the convention $L_n^0 = I$. Note that $L_n^m = 0$ for

$m \geq n + p$. Introduce

$$\phi_n = \sum_{m=0}^{n+p-1} \varphi_m L_n^m = \sum_{m=0}^{\infty} \varphi_m L_n^m$$

then

$$\phi_n^{-1} = \sum_{h=0}^{\infty} \alpha_h L_n^h = \sum_{h=0}^{n+p-1} \alpha_h L_n^h.$$

Set $x'_n = [x_{1-p}, \dots, x_n]$, $y'_n = [y_{1-p}, \dots, y_n]$ then $y_n = \phi_n x_n$.

With $\|L_n\| \leq 1$ this gives

$$\sum_{1-p}^n y_t y'_t = y'_n y_n = x'_n \phi'_n \phi_n x_n \leq \|\phi_n\|^2 x'_n x_n \leq \left\{ \sum_0^{\infty} |\phi_m| \right\}^2 \sum_{1-p}^n x_t x'_t$$

and, in the same way,

$$\sum_{1-p}^n x_t x'_t \leq \|\phi_n^{-1}\|^2 \cdot y'_n y_n \leq \left\{ \sum_0^{\infty} |\alpha_h| \right\}^2 \sum_{1-p}^n y_t y'_t.$$

This completes the proof.

Lemma 3.5. Let $(x_t)_{t \in \mathbb{N}}$ be a nonstochastic sequence of $k \times 1$ vectors.

a) Let $S_n := \sum_1^n x_t x'_t$ with $S_n > 0$ for some n . We have

$$\max_{(1 \leq t \leq n)} x'_t S_n^{-1} x_t \Rightarrow S_n^{-1} \rightarrow 0, \quad x'_n S_n^{-1} x_n \rightarrow 0$$

b) Let Z_n be a sequence of nonstochastic $k \times k$ matrices with $0 \leq Z_1 \leq Z_2 \leq \dots$ and $Z_n > 0$ for some n . We have

$$Z_n^{-1} \rightarrow 0, \quad x'_n Z_n^{-1} x_n \rightarrow 0 \Rightarrow \max_{(1 \leq t \leq n)} x'_t Z_n^{-1} x_t \rightarrow 0$$

Proof. a) Take some fixed $c \in \mathbb{R}_k$. Let N be such that $S_N > 0$. Then $c = \sum_1^N \alpha_t x_t$ for some $\alpha_1, \dots, \alpha_N$ not depending on n . Hence, for $n \geq N$:

$$c' S_n^{-1} c \leq \sum_1^N \sum_1^N |\alpha_t \alpha_s| |x'_t S_n^{-1} x_s| \leq \left\{ \sum_1^N |\alpha_t| \right\}^2 \cdot \max_{(1 \leq t \leq n)} x'_t S_n^{-1} x_t \rightarrow 0$$

This holds for any c , implying $S_n^{-1} \rightarrow 0$.

b) For given n let τ_n be the largest index for which the maximum is attained. If (τ_n) is bounded then (x_{τ_n}) is bounded. So,

$$x'_{\tau_n} Z_n^{-1} x_{\tau_n} \leq |x_{\tau_n}|^2 \cdot \|Z_n^{-1}\| \rightarrow 0.$$

If (τ_n) is unbounded then $\tau_n \uparrow \infty$. Since $\tau_n \leq n$ we have $Z_{\tau_n}^{-1} \geq Z_n^{-1}$. So,

$$x_{\tau_n} Z_n^{-1} x_{\tau_n} \leq x_{\tau_n}' Z_{\tau_n}^{-1} x_{\tau_n} \rightarrow 0.$$

Proof of theorem 2.2. The condition (2.2) is fulfilled. So we have only to prove that (2.5) - (2.7) imply (2.3) - (2.4).

From (2.6) and $s_n \leq n$ we get

$$(3.15) \quad \|Z_n^{-1}\| = O(s_n^{-1})$$

In particular $Z_n^{-1} \rightarrow 0$. With (2.7) and lemma 3.5b this leads to

$$(3.16) \quad \max_{(1 \leq t \leq n)} |Z_n^{-1/2} v_t| \rightarrow 0$$

We derive a suitable expression for x_t . Set $y_t = 0$ if $t \leq -p$, $w_t = 0$ if $t \leq 0$. Define ϵ_t for $t \leq 0$ by the relation (1.2). Then this relation holds for all $t \in \mathbb{Z}$. In particular $\epsilon_t = 0$ if $t \leq -p$. Let $\varphi(z) = 1/A(z) = \sum_{m=0}^{\infty} \varphi_m z^m$.

Then $\varphi(z)$ has convergence radius larger than 1 and $\varphi(z) \neq 0$ if $|z| \leq 1$. In particular $\sum_{m=0}^{\infty} |\varphi_m| < \infty$. The solution of (1.2) can be written as

$$y_t = \sum_{h=0}^{\infty} \varphi_h (\gamma' w_{t-h} + \epsilon_{t-h}), \quad t \in \mathbb{Z}.$$

Furthermore, since $\sum_{j=0}^k \varphi_j \alpha_{k-j} = \delta_{0k}$, we have

$$w_t = \sum_{j=0}^{\infty} \left(\sum_{k=0}^j \varphi_k \alpha_{j-k} \right) w_{t-j} = \sum_{k=0}^{\infty} \left(\sum_{j=k}^{\infty} \varphi_k \alpha_{j-k} \right) w_{t-j} = \sum_{k=0}^{\infty} \varphi_k \left(\sum_{j=0}^p \alpha_j w_{t-j-k} \right)$$

Therefore,

$$\begin{aligned} x_t' &= (w_t', y_{t-1}, \dots, y_{t-p}) = \\ &= \sum_{k=0}^{\infty} \varphi_k \left[\left(\sum_{j=0}^p \alpha_j w_{t-k-j}, \gamma' w_{t-k-1}, \dots, \gamma' w_{t-k-p} \right) + \right. \\ &\quad \left. + (0, \epsilon_{t-k-1}, \dots, \epsilon_{t-k-p}) \right]. \end{aligned}$$

This gives

$$(3.17) \quad x_t = \mu_t + \zeta_t = \sum_0^{\infty} \varphi_k (v_{t-k} + \eta_{t-k}), \quad t \in \mathbb{Z},$$

where $\eta'_t = (0, \varepsilon_{t-1}, \dots, \varepsilon_{t-p})$ and

$$\mu_t = \sum_0^{\infty} \varphi_k v_{t-k}, \quad \zeta_t = \sum_0^{\infty} \varphi_k \eta_{t-k}, \quad t \in \mathbb{Z}.$$

With (3.17) we can derive a lower bound for \bar{S}_n . From lemma 3.4 it follows that

$$\sum_{1-p}^n x_t x'_t \geq c_1 \sum_{p+1}^n (v_t + \eta_t) (v_t + \eta_t)'.$$

Since

$$\sum_{p+1}^n E\{\eta_t \eta'_t\} = \text{diag} (0, \sum_{p+1}^n \sigma_{t-1}^2, \dots, \sum_{p+1}^n \sigma_{t-p}^2) \geq (n-p)1 \cdot I_0$$

we get, by taking expectations,

$$\sum_{1-p}^0 E\{x_t x'_t\} + \bar{S}_n \geq c' Z_n$$

for some $c' > 0$. Since $Z_n^{-1} \rightarrow 0$ this implies

$$(3.18) \quad \bar{S}_n \geq c Z_n$$

for some $c > 0$.

With these preparations we can prove (2.3), (2.4). First we prove (2.4). From (3.17) and (3.18) we get

$$\begin{aligned} (c \max x'_t \bar{S}_n^{-1} x_t)^{1/2} &\leq \{\max x'_t Z_n^{-1} x_t\}^{1/2} = \max |Z_n^{-1/2} x_t| \\ &\leq \max |Z_n^{-1/2} \mu_t| + \max |Z_n^{-1/2} \zeta_t| \\ &\leq \sum_0^{\infty} |\varphi_k| \cdot (\max |Z_n^{-1/2} v_t| + \max |Z_n^{-1/2} \eta_t|). \end{aligned}$$

According to (3.16) the first term on the right-hand side tends to 0, and with (3.15) and (3.7) we see that the second term on the right-hand side tends to 0 in probability. Together this proves (2.4).

Finally we prove (2.3). From (3.15) and (3.17) we get

$$\begin{aligned}
 & \| \bar{S}_n^{-\frac{1}{2}} (\sum_1^n \lambda_t x_t x'_t - E\{\sum_1^n \lambda_t x_t x'_t\}) \bar{S}_n^{-\frac{1}{2}} \| \leq \\
 & \leq \| \bar{S}_n^{-\frac{1}{2}} Z_n^{\frac{1}{2}} \|^2 \cdot \| Z_n^{-\frac{1}{2}} (\sum_1^n \lambda_t x_t x'_t - E\{\sum_1^n \lambda_t x_t x'_t\}) Z_n^{-\frac{1}{2}} \| \\
 & \leq \| Z_n^{\frac{1}{2}} \bar{S}_n^{-1} Z_n^{\frac{1}{2}} \| \cdot \{ \| Z_n^{-\frac{1}{2}} (\sum_1^n \lambda_t \zeta_t \zeta'_t - E\{\sum_1^n \lambda_t \zeta_t \zeta'_t\}) Z_n^{-\frac{1}{2}} \| + \\
 & \quad + 2 \| Z_n^{-\frac{1}{2}} (\sum_1^n \lambda_t \mu_t \zeta'_t - E\{\sum_1^n \lambda_t \mu_t \zeta'_t\}) Z_n^{-\frac{1}{2}} \| \} \\
 & \leq c' \cdot s_n^{-1} \| \sum_1^n \lambda_t \zeta_t \zeta'_t - E\{\sum_1^n \lambda_t \zeta_t \zeta'_t\} \| + \\
 & \quad + c'' \cdot s_n^{-\frac{1}{2}} \| Z_n^{-\frac{1}{2}} (\sum_1^n \lambda_t \mu_t \zeta'_t - E\{\sum_1^n \lambda_t \mu_t \zeta'_t\}) \|
 \end{aligned}$$

for some constants c' , c'' . So it suffices to prove that

$$(3.19) \quad s_n^{-1} (\sum_1^n \lambda_t \xi_{t-i} \xi_{t-j} - E\{\sum_1^n \lambda_t \xi_{t-i} \xi_{t-j}\}) \xrightarrow{P} 0, \quad i, j \in \mathbb{Z}$$

$$(3.20) \quad s_n^{-\frac{1}{2}} \| Z_n^{-\frac{1}{2}} (\sum_1^n \lambda_t \mu_t \xi_{t-i} - E\{\sum_1^n \lambda_t \mu_t \xi_{t-i}\}) \| \xrightarrow{P} 0, \quad i \in \mathbb{Z},$$

where

$$\xi_t = \sum_0^\infty \varphi_h \varepsilon_{t-h}$$

The relation (3.19) follows from lemma 3.2 and the fact that (λ_t) is bounded. For the proof of (3.20) we write

$$(3.21) \quad \sum_1^n \lambda_t \gamma_t \xi_{t-i} = \sum_1^n \lambda_t \mu_t (\sum_0^\infty \varphi_h \varepsilon_{t-h-i}) = \sum_j a_{nj} \varepsilon_{j-i}$$

where

$$a_{nj} = \sum_1^n \lambda_t \mu_t \varphi_{t-j}$$

From (3.16) we get for some $c > 0$ that

$$|Z_n^{-\frac{1}{2}} a_{nj}| \leq \sum_0^\infty |\varphi_k| \cdot \max |Z_n^{-\frac{1}{2}} \mu_t| \leq c \cdot \max |Z_n^{-\frac{1}{2}} v_t| \rightarrow 0$$

and so

At first we derive some properties of P_n and Q_n . We have

$$P_n = \phi_{00}^D \phi_{n0}' + \phi_{01}^D \phi_{n1}' \geq \phi_{00}^D \phi_{n0}'$$

and so, with (2.8), (2.9) it follows that

$$(3.24) \quad \|P_n^{-1}\| = O(1/n).$$

Furthermore,

$$Q_n = \phi_{00}^D \phi_{n0}' + \phi_{01}^D \phi_{n1}'$$

and with (2.8) this gives

$$\begin{aligned} \|P_n^{-1} Q_n\| &\leq \|(\phi_{00}^D \phi_{n0}')^{-1} Q_n\| = \\ &= \|(\phi_{00}^{-1})' (\phi_{10}' + D_{n0}^{-1} \phi_{00}^{-1} \phi_{01}^D \phi_{n1}')\|. \end{aligned}$$

Using (2.10) this gives

$$(3.25) \quad \|P_n^{-1} Q_n\| = O(1).$$

Next we derive a lower bound for Z_n^{-1} . We have for some constants $c, c' > 0$:

$$\begin{aligned} Z_n &= \sum_{p+1}^n v_t v_t' + nI_0 = \phi D_n^{\frac{1}{2}} \left(\sum_{p+1}^n \tilde{q}_t(n) \tilde{q}_t'(n) \right) D_n^{\frac{1}{2}} \phi' + nI_0 \\ &\geq c' \phi D_n \phi' + nI_0 \end{aligned}$$

or

$$(3.26) \quad Z_n \geq c A_n$$

where

$$A_n = \begin{bmatrix} P_n & Q_n \\ Q_n' & R_n + nI \end{bmatrix}$$

Note that

$$A_n^{-1} = \begin{bmatrix} P_n^{-1} + T_n \Delta_n^{-1} T_n' & -T_n \Delta_n^{-1} \\ -\Delta_n^{-1} T_n' & \Delta_n^{-1} \end{bmatrix}$$

where $T_n = P_n^{-1} Q_n$, $\Delta_n = nI + R_n - Q_n' P_n^{-1} Q_n$. Since $\phi D_n \phi' \geq 0$ we have

$R_n - Q_n' P_n^{-1} Q_n \geq 0$ and so $\|\Delta_n^{-1}\| = O(1/n)$. Then from (3.24), (3.25) it follows that $\|A_n^{-1}\| = O(1/n)$ and with (3.26) this proves (2.6).

Finally, with (3.26) we get

$$\begin{aligned} v_z' Z_n^{-1} v_n &= \tilde{q}_n'(n) \cdot D_n^{\frac{1}{2}} \phi' Z_n^{-1} \phi D_n^{\frac{1}{2}} \tilde{q}_n(n) \\ &\leq \frac{1}{c} |\tilde{q}_n(n)|^2 \cdot \|D_n^{\frac{1}{2}} \phi' A_n^{-1} \phi D_n^{\frac{1}{2}}\| \\ &\leq \frac{1}{c} |\tilde{q}_n(n)|^2 \cdot \text{tr}(A_n^{-1} \cdot \phi D_n \phi') . \end{aligned}$$

Therefore (2.7) follows from (2.12) if we can prove that $\text{tr}(A_n^{-1} \phi D_n \phi') = O(1)$. However, this follows from

$$\begin{aligned} \text{tr}(A_n^{-1} \phi D_n \phi') &= \text{tr}\{(P_n^{-1} + T_n \Delta_n^{-1} T_n') P_n - T_n \Delta_n^{-1} Q_n'\} + \\ &\quad + \text{tr}\{-\Delta_n^{-1} T_n' Q_n + \Delta_n^{-1} R_n\} = \text{tr}\{I + \Delta_n^{-1} (R_n - Q_n' P_n^{-1} Q_n)\} = \\ &= p + \text{tr}\{\Delta_n^{-\frac{1}{2}} (R_n - Q_n' P_n^{-1} Q_n) \Delta_n^{-\frac{1}{2}}\} \leq 2p + q - r , \end{aligned}$$

completing the proof.

4. Weak consistency

We make some final remarks about weak consistency of the LS-estimator in the autoregressive model (1.2).

The conditions of theorem 2.2 imply $b_n \xrightarrow{P} \beta$. This follows from a short inspection of the proofs of theorems 2.1 and 2.2. The condition (2.6) implies $Z_n^{-1} \rightarrow 0$ and therefore also $C_n \rightarrow 0$. Then the convergence in distribution of $C_n^{-1/2}(b_n - \beta)$ gives $b_n \xrightarrow{P} \beta$.

In fact, for weak consistency the conditions of theorem 2.2 can be weakened considerably because we need only to prove that $\bar{S}_n^{-1} S_n \xrightarrow{P} I$ and $C_n \rightarrow 0$.

This can be done under the stability assumption (2.5), the assumption that the ϵ_t satisfy the LF condition instead of the Eicker conditions (see (3.6)), and the assumption $\|Z_n^{-1}\| = O(s_n^{-1})$, where in the definition of Z_n the term nI_0 is replaced by $s_n I_0$. The behaviour of Z_n can be investigated as in theorem 2.3. For other conditions about weak consistency, see e.g. Drygas [2], remark 4.5, and Eicker [3], Satz 1.

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